Abstract

Double operator integrals, introduced by Yu.L. Daletskii and S.G. Krein, play a fundamental role in the analysis of operator functions. Originating from the problem of differentiating operator-valued functions of the form $t \mapsto f(A + tK)$, these integrals provide an elegant framework for expressing derivatives in terms of spectral measures. A key result in this theory is the formula

$$\frac{d}{dt} f(A+tK)\Big|_{t=0} = \iint_{\mathbb{R}\times\mathbb{R}} \frac{f(x)-f(y)}{x-y}\, dE_A(x) K \, dE_A(y),$$

which holds for sufficiently regular functions f and highlights the role of double operator integrals in spectral perturbation theory.

Later developments by B.S. Birman and M.Z. Solomyak extended the scope of double operator integration, revealing its deep connections with perturbation theory and operator Lipschitz functions. A significant breakthrough in this area was achieved by D. Potapov and F. Sukochev in 2011, resolving a long-standing question posed by M.G. Krein in 1964. Their approach, based on harmonic analysis in UMD (unconditional martingale differences) Banach spaces, led to the best possible bound for the Schatten norm of $f(A) - f(B)$ in terms of $A - B$:

$$\|f(A) - f(B)\|_p \leq c_p \|f\|_{\mathrm{Lip}} \|A - B\|_p, \quad 1 < p < \infty.$$

This fundamental estimate applies to all scalar Lipschitz functions f and has profound implications for spectral shift functions, trace formulae in non-commutative analysis and non-commutative geometry.

In this talk, we will explore the foundational aspects of double operator integrals, their impact on perturbation theory, and their role in characterizing operator Lipschitz functions. We will also delve into the theory of higher-order operator differentiation and multiple operator integrals, uncovering powerful analytical tools that extend these ideas to more intricate settings.