A STUDY ON EIGENVALUE PERTURBATION THEOREMS

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Abstract. Perturbation of eigenvalues of complex matrices is a well studied concept in matrix analysis and has a rich history. One of the classical approaches is to study the spectral variation of one matrix with respect to the other matrix, which is defined as

$$sv_A(B) := \max_{1\leq j\leq n} \min_{1\leq i\leq n} |\mu_j - \lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ denote the eigenvalues of A and B respectively. A well-known bound for the spectral variation due to Elsner states that

$$sv_A(B) \leq ||A - B||_2^{1/n} \left(||A||_2 + ||B||_2\right)^{1-1/n}.$$

This was followed by several other results concerning spectral variation and other notions. A prominent result in this direction is the Hoffman-Wielandt inequality, which says the following: Let $A, B \in M_n(\mathbb{C})$ be normal matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$, respectively given in some order. Then there exists a permutation π on $\{1, \ldots, n\}$ such that

$$\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^2 \leq ||A - B||_F^2.$$

Several generalizations of the Hoffman-Wielandt inequality were studied by relaxing the normality condition on one or both of the matrices. One such generalization given by Ji. G. Sun, states the following: Let $A \in M_n(\mathbb{C})$ be a diagonalizable matrix and $B \in M_n(\mathbb{C})$ be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$, respectively. Then, there exists a permutation π on $\{1, \ldots, n\}$ such that

$$\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^2 \leq ||X||_2^2 ||X^{-1}||_2^2 ||A - B||_F^2,$$

where X is a nonsingular matrix whose columns are eigenvectors of A. Various eigenvalue perturbation results have been studied in the literature for nonnormal matrices, such as the Bauer - Fike theorem, which states the following: Let A and E be square matrices with A diagonalizable and B arbitrary. If μ is an eigenvalue of $A + E$, then there exists an eigenvalue λ of A such that

$$|\mu - \lambda| \leq ||P|| ||P^{-1}|| ||E||,$$

where P is a matrix that diagonalizes A and $|| \cdot ||$ is any matrix norm induced by an absolute norm.

Our aim is to study systematically the Hoffman-Wielandt inequality and its generalization for (i) standard eigenvalues of quaternion matrices, (ii) complex matrix polynomials and quaternion matrix polynomials and (iii) coneigenvalues of quaternion matrices. We also establish results analogous to the Bauer-Fike theorem, Elsner's theorem and other localization results such as the Geršgorin theorem for coneigenvalues of quaternion matrices.

References

[1] Basavaraju, P., Hadimani, S. and Jayaraman, S., "Hoffman-Wielandt type inequality for block companion matrices of certain matrix polynomials", *Adv. Oper. Theory*, **8**(4) (2023), paper No. 65, 14.

[2] Basavaraju, P., Hadimani, S. and Jayaraman, S., "On coneigenvalues of quaternion matrices: location and perturbation", *Indian J. Pure Appl. Math.*, accepted for publication (Special issue of ICLAA 2023), *arXiv:2704.08932*.

[3] Basavaraju, P., Hadimani, S. and Jayaraman, S., "The Hoffman-Wielandt inequality for quaternion matrices and quaternion matrix polynomials", *arXiv:2302.07638* (under review).