Endpoint estimate of rough maximal singular integral operator

Abstract: Study of singular integral operators was initiated by Calderón and Zygmund in 1950's, which continues to be the central theme of research in Fourier analysis.

For $\Omega : \mathbb{S}^{d-1} \to \mathbb{C}$ define

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} f(x-y)\Omega(\frac{y}{|y|})|y|^{-d} dy.$$

If Ω satisfies some "smoothness" condition then T_Ω belongs to the well understood class of Calderón-Zygmund operators. T_Ω is said to be rough singular integral operator if there is no "smoothness" in Ω. For $\Omega \in L\log L(\mathbb{S}^{d-1})$ with zero average, Calderón and Zygmund proved the L^p boundedness of T_Ω for $1 < p < \infty$. The end point $p = 1$ was more elusive and remained opened for almost 30 years. For dimension 2, Christ and Hofmann (independently) proved that $T_\Omega : L^1 \to L^{1,\infty}$ for slightly stronger hypothesis $\Omega \in L^q(\mathbb{S}^{d-1}),\ q > 1$. Later Christ and Rubio de Francia proved this result for $\Omega \in L\log L(\mathbb{S}^{d-1})$ again for $d = 2$. Finally, Seeger in 1996 settled this for all dimensions and $\Omega \in L\log L(\mathbb{S}^{d-1})$. To study the pointwise existence of $T_\Omega f$ one considers the following maximal operator,

$$T_\Omega^* f(x) = \sup_{\epsilon>0} \left| \int_{|y|>\epsilon} f(x-y)\Omega(\frac{y}{|y|})|y|^{-d} dy \right|.$$

Duoandikoetxea and Rubio de Francia proved that $\|T_\Omega^* f\|_p \lesssim \|f\|_p,\ 1 < p < \infty$. After Seeger's result in 1996 it is an open problem to show $T_\Omega^* : L^1 \to L^{1,\infty}$. In this talk we will discuss about the best known result for boundedness of T_Ω^* near L^1.

This is a joint work with Ankit Bhojak.